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# COMPLETE NONCOMPACT THREE-MANIFOLDS WITH NONNEGATIVE RICCI CURVATURE

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## 1. Introduction

In this paper we are going to prove the following main theorem.

**Theorem 1.1.** *Suppose $M$ is a complete noncompact three-dimensional Riemannian manifold with bounded nonnegative Ricci curvature. Then $M$ is diffeomorphic to a quotient space of one of the spaces $\mathbf{R}^3$ or $S^2 \times \mathbf{R}^1$ by a group of fixed point free isometries in the standard metrics.*

The classification of three-dimensional Riemannian manifolds with nonnegative Ricci curvature has been an interesting problem for many years. In the noncompact case, R. Schoen and S. T. Yau [3] proved that every complete noncompact three-dimensional Riemannian manifold with positive Ricci curvature is diffeomorphic to $\mathbf{R}^3$; if the Ricci curvature is only nonnegative, the problem is still open.

On the other hand, R. S. Hamilton [1] developed the important heat equation method to deal with the compact three-dimensional Riemannian manifolds with positive Ricci curvature and proved that such a manifold is diffeomorphic to the quotient space of $S^3$. Using his argument, finally in 1986 Hamilton [2] gave an entire classification of compact three-dimensional Riemannian manifolds with nonnegative Ricci curvature.

Theoretically speaking, one can use the heat equation method to classify the complete noncompact three-dimensional Riemannian manifolds with nonnegative Ricci curvature just exactly the same way as Hamilton did in the compact case in [1] and [2]. But we still have some technical problems which come from the noncompactness of the manifold; in the compact manifold case the heat equation always has a solution for at least a short time interval and this is not true in the noncompact manifold case.

If we assume the curvature of the noncompact manifold is bounded, then we have the short time existence for the solution of the heat equation on the manifold. The short time existence theorem for the heat equation was proved by author [4] under the assumption of bounded curvature tensor, thus we can

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prove Theorem 1.1 by using the same arguments as Hamilton used in [1] and [2].

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## 2. Preliminary results

For any $n$-dimensional Riemannian manifold $M$ with the metric

$$ds^2 = g_{ij}(x)\,dx^i\,dx^j > 0$$

we use $Rm = \{R_{ijkl}\}$ to denote the Riemannian curvature tensor and use

$$R_{ij} = g^{kl}R_{ikjl}, \qquad R = g^{ij}R_{ij} = g^{ij}g^{kl}R_{ikjl}$$

to denote the Ricci curvature and scalar curvature respectively, where $(g^{ij}) = (g_{ij})^{-1}$.

We use $|R_{ijkl}|^2$ to denote the norm of the curvature tensor and use $\nabla$ to denote the covariant derivative. For any integer $m > 0$, $\nabla^m R_{ijkl}$ denotes all of the $m$th order derivatives of the curvature tensor.

Consider the heat equation

$$\begin{aligned} &\frac{\partial}{\partial t} g_{ij}(x,t) = -2R_{ij}(x,t), \\ &g_{ij}(x,0) = g_{ij}(x) \end{aligned} \tag{1}$$

on the manifold. We have the following short time existence theorem in the noncompact case:

**Theorem 2.1.** *Let $M$ be an $n$-dimensional complete noncompact Riemannian manifold with its Riemannian curvature tensor $\{R_{ijkl}\}$ satisfying*

$$|R_{ijkl}|^2 \le \kappa_0 \quad \text{on } M, \tag{2}$$

*where $0 < \kappa_0 < +\infty$ is a constant. Then there exist constants $T = T(n,\kappa_0) > 0$ and $C_m = C_m(n,\kappa_0) > 0$ depending only on $n$ and $\kappa_0$ for $m = 0,1,2,3,\ldots$ such that the evolution equation (1) has a smooth solution $g_{ij}(x,t) > 0$ for a short time $0 \le t \le T$, and satisfies*

$$\sup_{x\in M} |\nabla^m R_{ijkl}(x,t)|^2 \le C_m/t^m, \qquad 0 \le t \le T, \tag{3}$$

*for all integers $m \ge 0$.*

*Proof.* This is Theorem 1.1 in [4].

In the remainder of this paper, we always assume that $M$ is a complete noncompact three-dimensional Riemannian manifold with its Ricci curvature $\{R_{ijkl}\}$ satisfying

$$0 \le \{R_{ij}\} \le \kappa_0 g_{ij} \quad \text{on } M, \tag{4}$$

where $0 < \kappa_0 < +\infty$ is some constant.

**Theorem 2.2.** *In dimension three we have*

$$R_{ijkl} = g_{ik}R_{jl} - g_{il}R_{jk} - g_{jk}R_{il} + g_{jl}R_{ik} - \tfrac{1}{2}R(g_{ik}g_{jl} - g_{il}g_{jk}). \tag{5}$$

*Proof.* This is Theorem 8.1 in [1].

This result means that we can control the full Riemannian curvature tensor $R_{ijkl}$ just from the Ricci curvature $R_{ij}$ in the three-dimensional manifold. Thus if (4) is satisfied, using (5) we get

$$|R_{ijkl}|^2 \le 4000\kappa_0^2 \quad \text{on } M. \tag{6}$$

Combining (6) and Theorem 2.1 we have

**Corollary 2.3.** *Suppose $M$ is a complete noncompact three-dimensional Riemannian manifold with its Ricci curvature satisfying*

$$0 \le R_{ij} \le \kappa_0 g_{ij} \quad on\ M,$$

*where $0 < \kappa_0 < +\infty$ is some constant. Then there exist constants $T = T(\kappa_0) > 0$ and $C_m = C_m(\kappa_0) > 0$ depending only on $\kappa_0$ for $m = 0, 1, 2, 3, \ldots$ such that the evolution equation* (1) *has a smooth solution $g_{ij}(x,t) > 0$ on $M \times [0,T]$, and satisfies*

$$\operatorname*{Sup}_{x\in M} |\nabla^m R_{ijkl}(x,t)|^2 \le C_m/t^m, \qquad 0 \le t \le T, \tag{7}$$

*for all integers $m \ge 0$.*

Moreover, we have the following lemma.

**Lemma 2.4.** *Suppose $g_{ij}(x,t) > 0$ defined on $M \times [0,T]$ is the solution obtained in Corollary* 2.3. *Then*

$$0 \le R_{ij}(x,t) \le \sqrt{3C_0}\, g_{ij}(x,t) \tag{8}$$

*holds for all $(x,t) \in M \times [0,T]$, where $0 < C_0 < +\infty$ is the constant in* (7).

*Proof.* From (7) we know that

$$|R_{ijkl}(x,t)|^2 \le C_0, \qquad (x,t) \in M \times [0,T]. \tag{9}$$

Thus we have

$$|R_{ij}(x,t)|^2 \le 3C_0 \quad \text{on } M \times [0,T],$$

$$-\sqrt{3C_0}\, g_{ij}(x,t) \le R_{ij}(x,t) \le \sqrt{3C_0}\, g_{ij}(x,t) \tag{10}$$

for all $(x,t) \in M \times [0,T]$.

Just the same as R. S. Hamilton did in [2], we pick an abstract vector bundle $V$ isomorphic to the tangent bundle $TM$, but with a fixed metric $h_{ab}$ on the fibers. We choose an isometry $U = \{U_a^i\}$ between $V$ and $TM$ at time $t = 0$, and let the isometry $U$ evolve by the equation

$$\frac{\partial}{\partial t} U_a^i = g^{ij} R_{jk} U_a^k. \tag{11}$$

Then the pull-back metric

$$h_{ab} = g_{ij} U_a^i U_b^j \tag{12}$$

remains constant in time. It is easy to see that $\frac{\partial}{\partial t} h_{ab} \equiv 0$ and $U$ remains an isometry between the varying metric $g_{ij}$ on $TM$ and the fixed metric $h_{ab}$ on $V$. We use $U$ to pull back the curvature tensor to get a tensor on $V$:

$$R_{abcd} = R_{ijkl} U_a^i U_b^j U_c^k U_d^l. \tag{13}$$

We can also pull back the Levi-Civita connection $\Gamma = \{\Gamma_{ij}^k\}$ on $M$ to get a connection $\widetilde{\Gamma} = \{\widetilde{\Gamma}_{jc}^a\}$ on $V$, where the covariant derivative of a section $w = \{w^a\}$ of $V$ is given locally by

$$\nabla_i w^a = \frac{\partial w^a}{\partial x^i} + \widetilde{\Gamma}_{ib}^a w^b. \tag{14}$$

Thus we may take the covariant derivative of any tensor of $V$ and $TM$. In particular we have

$$\nabla_i U_a^j = 0, \qquad \nabla_i h_{ab} = 0. \tag{15}$$

We define the Laplacian operator

$$\Delta R_{abcd} = g^{ij} \nabla_i \nabla_j R_{abcd} \tag{16}$$

to be the trace of the second covariant derivatives. Then from Hamilton [2] we have

$$\frac{\partial}{\partial t} R_{abcd} = \Delta R_{abcd} + 2(B_{abcd} - B_{abdc} - B_{adbc} + B_{acbd}), \tag{17}$$

where $B_{abcd} = R_{aebf} R_{cedf}$.

Using (17) it is easy to see that

$$\frac{\partial}{\partial t} R_{ab} = \Delta R_{ab} + Q_{ab}, \tag{18}$$

where $Q_{ab}$ is quadratic in $R_{ab}$. If $R_{ab}$ is diagonal:

$$(R_{ab}) = \begin{pmatrix} \lambda & & \\ & \mu & \\ & & \nu \end{pmatrix}, \tag{19}$$

then $Q_{ab}$ is also diagonal:

$$(Q_{ab}) = \begin{pmatrix} \rho & & \\ & \sigma & \\ & & \tau \end{pmatrix}, \tag{20}$$

where

$$\begin{aligned} \rho &= (\mu - \nu)^2 + \lambda(\mu + \nu), \\ \sigma &= (\lambda - \nu)^2 + \mu(\lambda + \nu), \\ \tau &= (\lambda - \mu)^2 \nu(\lambda + \mu). \end{aligned} \tag{21}$$

Thus

$$(Q_{ab}) \geq 0 \quad \text{if } (R_{ab}) \geq 0. \tag{22}$$

Because $R_{ab} = R_{ij} U_a^i U_b^j$, from (10) we have

$$-\sqrt{3C_0} h_{ab}(x,t) \leq R_{ab}(x,t) \leq \sqrt{3C_0} h_{ab}(x,t) \tag{23}$$

on $M \times [0,T]$. By assumption, $R_{ij}(x,0) \geq 0$; we have

$$R_{ab}(x,0) \geq 0 \quad \text{on } M. \tag{24}$$

From (18), (22), (23), (24) and using the cut-off function argument as we did in the proof of Theorem 4.14 of [5] we know that

$$R_{ab}(x,t) \geq 0 \quad \text{on } V \times [0,T]. \tag{25}$$

This implies

$$R_{ij}(x,t) \geq 0 \quad \text{on } M \times [0,T]. \tag{26}$$

From (10) and (26) we know that Lemma 2.4 is true.

Thus the nonnegativity of the Ricci curvature is preserved by the heat equation in the case of dimension three. If the Ricci curvature of $M$ becomes strictly positive after a short time, then we can apply the following theorem.

**Theorem 2.5.** *Let $M$ be a complete noncompact three-dimensional Riemannian manifold with positive Ricci curvature. Then $M$ is diffeomorphic to* $\mathbf{R}^3$.

*Proof.* This is Theorem 3 in [3].

## 3. Local decomposition

Using the notation of the last section, suppose $V$ is the abstract vector bundle isomorphic to the tangent bundle $TM$, $h_{ab}$ is the fixed metric on $V$, $U = \{U_a^i\}$ is the isometry between $V$ and $TM$, and $R_{ab} = R_{ij} U_a^i U_b^j$.

We can regard $(R_{ab})$ as a symmetric bilinear form on $V$; from (18), (22), (23), and (25) of §2 we know that

$$\begin{aligned} &\frac{\partial}{\partial t} R_{ab} = \Delta R_{ab} + Q_{ab}, \\ &Q_{ab} \geq 0 \quad \text{if } R_{ab} \geq 0, \end{aligned} \tag{1}$$

$$0 \leq R_{ab} \leq \sqrt{3C_0} h_{ab} \quad \text{on } V \times [0,T]. \tag{2}$$

If the Ricci curvature of $M$ does not become strictly positive after a short time, we have the following lemma.

**Lemma 3.1.** *There exists a constant $\delta > 0$ such that on the time interval $0 < t < \delta$ the rank of $(R_{ab})$ is constant and the null space of $(R_{ab})$ is invariant under parallel translation and invariant in time and also lies in the null space of $(Q_{ab})$.*

*Proof.* The proof of this lemma is exactly the same as the proof of Lemma 8.2 in [2]. Because all of the arguments used in that proof are local arguments, they also work in the noncompact case.

Now we are in the position to prove Theorem 1.1. We know that on $0 < t < \delta$, the rank of $(R_{ab})$ is constant and is invariant in time because the null space of $(R_{ab})$ is invariant in time.

*Case* A. $\operatorname{rank}(R_{ab}) = 0$ on $V \times (0, \delta)$.

In this case $R_{ab} \equiv 0$; thus

$$R_{ij}(x,t) \equiv 0 \quad \text{on } M \times (0, \delta). \tag{3}$$

We know that in this case $M$ is diffeomorphic to a quotient space of $\mathbf{R}^3$ by a group of fixed point free isometries in the standard metric.

*Case* B. $\operatorname{rank}(R_{ab}) = 3$ on $V \times (0, \delta)$.

In this case $R_{ab} > 0$ on $V \times (0, \delta)$; thus

$$R_{ij}(x,t) > 0 \quad \text{on} M \times (0, \delta). \tag{4}$$

Using Theorem 2.5 we know that $M$ is diffeomorphic to $\mathbf{R}^3$.

*Case* C. $\operatorname{rank}(R_{ab}) = 1$ on $V \times (0, \delta)$.

We can write $(R_{ab})$ as

$$(R_{ab}) = \begin{pmatrix} \lambda & & \\ & 0 & \\ & & 0 \end{pmatrix}, \qquad \lambda > 0.$$

Using (20) and (21) of §2 we have

$$(Q_{ab}) = \begin{pmatrix} 0 & & \\ & \lambda^2 & \\ & & \lambda^2 \end{pmatrix}.$$

Thus the null space of $(R_{ab})$ does not lie in the null space of $(Q_{ab})$; from Lemma 3.1 we know that is impossible.

*Case* D. $\operatorname{rank}(R_{ab}) = 2$ on $V \times (0, \delta)$.

Fix a time $t_0 \in (0, \delta)$. The null space of $(R_{ab})$ is spanned by a translation-invariant vector field $V^k$ on $V$. Using $U = \{U_a^i\}$ to pull back to $TM$, we know that the null space of $(R_{ij})$ is spanned by a translation-invariant vector field $V^k$ on $TM$. Thus the tangent bundle $TM$ has an orthogonal decomposition

$$TM = V_1 \oplus V_2, \tag{5}$$

where $V_1$ and $V_2$ are invariant under parallel translation. $V_1$ is the null space of $(R_{ij})$, $\dim V_1 = 1$, $\dim V_2 = 2$.

**Lemma 3.2.** *If the tangent bundle $TM$ has an orthogonal decomposition $TM = V_1 \oplus V_2$, where $V_1$ and $V_2$ are invariant under parallel translation, then locally there is a product decomposition $M = M_1 \times M_2$ such that the metric on $M$ is the product of metrices on $M_1$ and $M_2$ and $V_1 = TM_1$, $V_2 = TM_2$.*

*Proof.* This is a lemma in §9 of [2].

Therefore the manifold $M$ splits locally as a product

$$M = \mathbf{R}^1 \times M^2, \tag{6}$$

where $M^2$ is a surface of positive curvature and $\mathbf{R}^1$ is flat. It is easy to see that such a local decomposition is always unique. For each leaf $M^2$ in (6), we extend $M^2$ as much as possible on $M$ to get a maximal leaf. For each point $x \in M$, we use $M^2_x$ to denote the maximal leaf passing through $x$. Because of the uniqueness of the local decomposition (6), we know that for any $x \in M$, $M^2_x$ is a complete surface of positive curvature and without boundary. The following properties hold:

$$M = \bigcup_{x \in M} M^2_x, \tag{7}$$

$$M^2_x = M^2_y \quad \text{if } M^2_x \cap M^2_y \neq \varnothing. \tag{8}$$

Fix a point $p \in M$. Since the maximal leaf $M^2_p$ is a complete surface of positive curvature and without boundary, we know that $M^2_p$ is diffeomorphic to $\mathbf{R}P^2$, $S^2$ or $\mathbf{R}^2$. If $M^2_p$ is diffeomorphic to $S^2$ or $\mathbf{R}^2$, then $M^2_p$ is an orientable surface. Using the local decomposition (6) we can find a covering map

$$\varphi \colon M^2_p \times \mathbf{R}^1 \to M, \tag{9}$$

where locally $\varphi$ is an isometry. Thus in this case the universal covering space $\widetilde{M}$ of $M$ is isometric to $\mathbf{R}^2 \times \mathbf{R}^1$ or $S^2 \times \mathbf{R}^1$, where $S^2$ or $\mathbf{R}^2$ have positive curvature and $\mathbf{R}^1$ is flat. If $M^2_p$ is diffeomorphic to $\mathbf{R}P^2$, then by the same reasoning as in (9) we can find a covering map

$$\varphi \colon S^2 \times \mathbf{R}^1 \to M. \tag{10}$$

From (6) we know that locally $\varphi$ is an isometry; thus $\widetilde{M}$ is isometric to $S^2 \times \mathbf{R}^1$ where $S^2$ has positive curvature and $\mathbf{R}^1$ is flat.

Thus in all the cases the universal covering space $\widetilde{M}$ of $M$ is isometric to $\mathbf{R}^2 \times \mathbf{R}^1$ or $S^2 \times \mathbf{R}^1$ with $S^2$ or $\mathbf{R}^2$ having positive curvature and $\mathbf{R}^1$ flat, and $M$ is a quotient space of $\mathbf{R}^2 \times \mathbf{R}^1$ or $S^2 \times \mathbf{R}^1$ by isometries. If the metric on $S^2$ is nonstandard or the metric on $\mathbf{R}^2$ is not flat, we can replace them by some conformally equivalent metrics with constant curvature or zero curvature. This will not change the group of isometries. Finally we know that $M$ is a quotient space of $\mathbf{R}^3$ or $S^2 \times \mathbf{R}^1$ by a group of isometries in the standard metrics.

Collecting the results of Cases A, B, C and D we know that Theorem 1.1 is true and this completes the proof of the main theorem.

**Remark.** We can generalize these arguments to complete noncompact four-dimensional Riemannian manifolds with nonnegative bounded curvature operator, just as [2] for the case of compact four-manifolds with nonnegative curvature operator.

Suppose $M$ is a complete noncompact four-dimensional Riemannian manifold with bounded nonnegative curvature operator. From Theorem 2.1 we know that the heat equation

$$\frac{\partial}{\partial t} g_{ij} = -2R_{ij} \tag{11}$$

has a smooth solution for a short time. Using Theorem 4.14 in [5] we know that the heat flow (11) preserves the nonnegativity of the curvature operator. Thus using the same arguments as Hamilton used in his paper [2] and the fact that every complete noncompact Riemannian manifold with positive curvature operator is diffeomorphic to $\mathbf{R}^n$ and the theorems we have proved in this paper we can finally get the following result:

**Theorem 3.3.** *Suppose $M$ is a complete noncompact four-dimensional Riemannian manifold with bounded nonnegative curvature operator. Then the universal covering $\widetilde{M}$ of $M$ is diffeomorphic to one of the spaces*

$$\mathbf{R}^4, \quad S^3 \times \mathbf{R}^1, \quad S^2 \times \mathbf{R}^2, \quad N,$$

*where $N$ is a Kähler surface with positive holomorphic bisectional curvature.*

We omit the details for the proof of this theorem; basically one can get the same result from J. Cheeger and D. Gromoll, *On the structure of complete manifolds of nonnegative curvature*, Ann. of Math. (2) **96** (1972) 413–443.

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